[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1013 C

Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : B.Sc. (Hons.) Mathematics CBCS (LOCF)

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any two parts from each question.

1. (a) Let (X, d) be a metric space. Define the mapping

$d^*: X \times X \to \mathbb{R}$ by

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)}; \ \forall x, y \in X.$$

Show that (X, d^*) is a metric space and $d^*(x, y) < 1$, for every $x, y \in X$. (6)

(b) Let $\langle x_n \rangle_{n\geq1}$ be a sequence of real numbers defined by $x_1 = a$, $x_2 = b$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for $n = 1, 2, \cdots$. Prove that $\langle x_n \rangle_{n\geq1}$ is a Cauchy sequence in $\mathbb{R}$ with usual metric. (6)

(c) Define a complete metric space. Is the metric space $(\mathbb{Z}, d)$ of integers, with usual metric d, a complete metric space? Justify. (6)

2. (a) (i) Let (X, d) be a metric space. Show that for every pair of distinct points x and y of X, there exist disjoint open sets U and V such that $x \in U$, $y \in V$. (2)

(ii) Give an example of the following :

(a) A set in a metric space which is neither a closed ball nor an open set. (1)

(b) A metric space in which the interior of the intersection of an arbitrary family of the subsets may not be equal to the intersection of the interiors of the members of the family. (2)

(c) A metric space in which every singleton is an open set. (1)

(b) Let (X, d) be a metric space. Let A be a subset of X. Define closure of A and show that it is the smallest closed superset of A. (6)

(c) Let (X, d) be a complete metric space. Let $\langle F_n \rangle$ be a nested sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Show that $\bigcap_{n=1}^{\infty} F_n$ is a singleton. Does it hold if (X, d) is incomplete? Justify. (6)

3. (a) Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$ be a function. Prove that f is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X. (6)

(b) Let A and B be non-empty disjoint closed subsets of a metric space (X, d). Show that there is a continuous real valued function f on X such that $f(x) = 0, \forall x \in A$, $f(x) = 1, \forall x \in B$ and $0 \leq f(x) \leq 1$, $\forall x \in X$. Further show that there exist disjoint open subsets G, H of X such that $A \subseteq G$ and $B \subseteq H$. (6)

(c) Define a dense subset of a metric space (X, d). Let $A \subseteq X$. Show that A is dense in X if and only if A^C has empty interior. Give an example of a metric space that has only one dense subset.

(6)

4. (a) Show that the metrics d_1, d_2 and d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| ,$$

$$d_2(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} \text{ and}$$

$$d_\infty(x,y) = \max\{ |x_i - y_i| : 1 \le i \le n\}$$

are equivalent where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. (6.5)

(b) Show that the function $f: \mathbb{R} \to (-1, 1)$ defined by

$f(x) = \frac{x}{1+|x|}$ is a homeomorphism but not an isometry. (6.5)

(c) (i) Let (X, d) be a complete metric space. Let $T: X \to X$ be a mapping such that $d(Tx,Ty) < d(x, y)$, $\forall\, x, y \in X$. Does T always have a fixed point? Justify. (4)

(ii) Let X be any non-empty set and $T: X \to X$ be a mapping such that T^n (where n is a natural number, $n > 1$) has a unique fixed point $x_0 \in X$. Show that x_0 is also a unique fixed point of T. (2.5)

5. (a) Let $(\mathbb{R}, d)$ be the space of real numbers with usual metric. Prove that a connected subset of E must be an interval. Give an example of two connected subsets of E, such that their union is disconnected. (4+2.5)

(b) Let (X, d) be a metric space such that every two points of X are contained in some connected subset of X. Show that (X, d) is connected.

(6.5)

(c) Let (X, d) be a metric space. Then prove that (X, d) is disconnected if and only if there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). (6.5)

6. (a) Prove that homeomorphism preserves compactness. Hence or otherwise show that

$$S(0,1) = \{z \in \mathbb{C} : |z| < 1\} \text{ and}$$

$$S[0,1] = \{z \in \mathbb{C} : |z| < 1\}$$

are not homeomorphic. (4+2.5)

(b) Let (X, d) be a metric space and $A \subseteq X$ such that every sequence in A has a subsequence converging in A. Show that for any $B \subseteq X$, there is a point $p \in A$ such that $d(p,B) = d(A,B)$. If B be a closed subset of X such that $A \cap B = \phi$, show that $d(A,B) > 0$. (4.5+2)

(c) Let f be a continuous real-valued function on a compact metric space (X, d_X), then show that f is bounded and attains its bounds. Does the result hold when X is not compact? Justify.

(4+2.5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1049 **C**

Unique Paper Code : 32351502

Name of the Paper : BMATH512: Group Theory-II

Name of the Course : **B.Sc. (H) Mathematics**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Question No. **1** has been divided into **10** parts and each part is of **1.5** marks.

4. Each question from Q. Nos. **2** to **6** has **3** parts and each part is of **6** marks. Attempt any **two** parts from each Question.

1. State true (T) or false (F). Justify your answer in brief.

 (i) A group of order p^2, p is a prime is always isomorphic to Z_{p2}.

(ii) A group of order 15 is always cyclic.

(iii) A group of order 14 is simple.

(iv) The smallest positive integer n such that there are two non-isomorphic groups of order n is 6.

(v) Every inner automorphism induced by an element 'a' of group G is an automorphism of G.

(vi) A abelian group of order 12 must have an element of order either 2 or 3.

(vii) U(105) is isomorphic to external direct product of U(21) and U(5).

(viii) Center of a group G is always a subgroup of normalizer of A in G, where A is any subset of G.

(ix) Aut(Z_{10}) is a cyclic group of automorphisms of G.

(x) The largest possible order for an element of $Z_{20} \oplus Z_{30}$ is 60.

2. (a) Define inner automorphism induced by an element 'a' of group G and find the group of all inner automorphisms of D_4.

(b) Defile the characteristic and the commutator subgroup of a group. Prove that the centre of a group is characteristic subgroup of the group.

(c) Let G' be the subgroup of commutators of a group G. Prove that G/G' is abelian. Also, prove that if G/N is abelian, then $N \geq G'$.

3. (a) Determine the number of cyclic subgroups of order 15 in $Z_{90} \oplus Z_{36}$.

(b) Define the internal direct product of the subgroups H and K of a group G. Prove that every group of order p^2, where p is a prime, is isomorphic to Zp^2 or $Zp \oplus Zp$ (external direct product of Zp with itself).

(c) Consider the group G = {1, 9, 16, 22, 29, 53, 74, 79, 81} under multiplication modulo 91. Determine the isomorphism class of G.

4. (a) Show that the additive group Z acts on itself by $z.a = z+a$ for all $z, a \in Z$.

(b) Show that an action is faithful if and only if its kernel is the identity subgroup.

(c) Let G be a group. Let H be a subgroup of G. Let G act by left multiplication on the set A of all left cosets of H in G. Let π_H be the permutation representation of G associated with this action. Prove that

(i) G acts transitively on A

(ii) The stabilizer of the point $1H \in A$ is the subgroup H.

(iii) $\text{Ker } \pi_H = \cap_{x \in G} xHx^{-1}$

5. (a) Let G be a permutation group on a set A (G is subgroup of S_A), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$, here G_x denotes stabilizer of x. Deduce that if G acts transitively on A then $\cap_{x \in G} \sigma G_a \sigma^{-1} = 1$.

(b) Show that every group of order 56 has a proper nontrivial normal subgroup.

(c) State Index theorem and prove that a group of order 80 is not simple.

6. (a) State the Class Equation for a finite group G. and use it to prove that p-groups have non trivial centers.

(b) Prove that group of order 255 is always cyclic.

(c) Show that the alternating group ~~As~~ A5 does not contain a subgroup of order 30, 20, or 15.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1132 C

Unique Paper Code : 32357501

Name of the Paper : DSE-I Numerical Analysis (LOCF)

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. All **six** questions are compulsory.

3. Attempt any two parts from each question.

4. Use of non-programmable scientific calculator is allowed.

P.T.O.

1. (a) Define fixed point of a function and construct an algorithm to implement the fixed point iteration scheme to find a fixed point of a function. Find the fixed point of $f(x) = 2x(1 - x)$. (6)

(b) Perform four iterations of Newton's Raphson method to find the positive square root of 18. Take initial approximation $x_0=4$. (6)

(c) Find the root of the equation $x^3 - 2x - 6= 0$ in the interval (2, 3) by the method of false position. Perform three iterations. (6)

2. (a) Define the order of convergence of an iterative method for finding an approximation to the root of $g(x) = 0$. Find the order of convergence of Newton's iterative formula. (6.5)

(b) Find a root of the equation $x^3 - 4x - 8 = 0$ in the interval (2, 3) using the Bisection method till fourth iteration. (6.5)

(c) Perform three iterations of secant method to determine the location of the approximate root of the equation $x^3 + x^2 - 3x - 3 = 0$ on the interval (1, 2). Given the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the approximations just obtained. (6.5)

3. (a) Using scaled partial pivoting during the factor step, find matrices L, U and P such that LU = PA

where $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{pmatrix}$ (6.5)

(b) Set up the SOR method with w=0.7 to solve the system of equations:

$$3x_1 - x_2 + x_3 = 4$$

$$2x_1 - 6x_2 + 3x_3 = -13$$

$$-9x_1 + 7x_2 - 20x_3 = 7$$

Take the initial approximation as $X^{(0)} = (0, 0, 0)$ and do three iterations. (6.5)

(c) Set up the Gauss-Jacobi iteration scheme to solve the system of equations:

$$10x_1 + x_2 + 4x_3 = 31$$

$$x_1 + 10x_2 - 5x_3 = -23$$

$$3x_1 - 2x_2 + 10x_3 = 38$$

Take the initial approximation as $X^{(0)} = (1, 1, 0)$ and do three iterations. (6.5)

4. (a) Obtain the piecewise linear interpolating polynomials for the function f(x) defined by the data:

x	1	2	4	8
f(x)	3	7	21	73

(6)

(b) Calculate the Newton second order divided difference $\frac{1}{x^2}$ of based on the points x_0, x_1, x_2. (6)

(c) Obtain the Lagrange form of the interpolating polynomial for the following data:

x	1	2	5
$f(x)$	-11	-23	1

(6)

P.T.O.

5. (a) Find the highest degree of the polynomial for which the second order backward difference approximation for the first derivative

$$f'(x_0) \approx \frac{3f(x_0) - 4f(x_0-h) + f(x_0-2h)}{2h}$$

provides the exact value of the derivative irrespective of h. (6)

(b) Derive second-order forward difference approximation to the first derivative of a function f given by

$$f'(x_0) \approx \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h}.$$

(6)

(c) Approximate the derivative of $f(x) = \sin x$ at $x_0 = \pi$ using the second order central difference formula taking $h = ½$, ¼ and 1/8 and then extrapolate from these values using Richardson extrapolation. (6)

6. (a) Using the Simpson's rule, approximate the value of the integral $\int_2^5 \ln x \, dx$. Verify that the theoretical error bound holds. (6.5)

(b) Apply Euler's method to approximate the solution of initial value problem $\frac{dx}{dt} = \frac{e^t}{x}$, $0 \le t \le 2$, $x(0) = 1$ and $N = 4$.

Given that the exact solution is $x(t) = \sqrt{2e^t - 1}$, compute the absolute error at each step. (6.5)

(c) Apply the optimal RK2 method to approximate the solution of the initial value problem $\frac{dx}{dt} = 1 + \frac{x}{t}$, $1 \le t \le 2$, $x(1) = 1$ taking the step size as $h = 0.5$. (6.5)